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LETTER TO THE EDITOR

Effective viscosity due to local turbulence interactions near the cutoff wavenumber in a constrained numerical simulation

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Abstract. If the largest resolved wavenumber in a numerical simulation of isotropic turbulence is too small, then it is well known that the energy spectrum will depart from its expected monotonic decrease with increasing wavenumber and will instead begin to increase. We show that an operational method is capable of modifying the instantaneous velocity field such that the unphysical features of the spectrum are suppressed. When the effect on the constrained simulation is interpreted in terms of an effective viscosity, this agrees well with the usual result obtained by comparison with a fully resolved direct numerical simulation, thus directly establishing the localness in wavenumber of the relevant interactions.

The numerical simulation of systems with many degrees of freedom has been of great interest in physics for some time and in practice there is often a requirement to reduce the number of degrees of freedom which have to be explicitly simulated. In fluid dynamics, the recognition that numerical simulation could be employed if one reduced the number of degrees of freedom goes back to meteorological work in the 1960s, when the idea of a large-eddy simulation (LES) was first put forward.

In applying the concept of LES to spectral simulations, the conventional approach is to resolve Fourier modes with wavenumbers up to some $k = K_C$, and to model the nonlinear transfer to the (now nonexistent) modes with $k \geqslant K_C$ by means of some additional viscosity acting on the resolved modes. In general, one may expect an effective viscosity of this kind to depend on both the local wavenumber k and on the cutoff wavenumber K_C. (Also one may expect some other nonlinear effects to be present, in addition to the Newtonian-type effective viscosity.) In this letter we introduce a novel method of compensating for the absence of nonlinear transfers, which also establishes the localness in wavenumber of the interactions.

We study the numerical simulation of stationary, homogeneous, isotropic turbulence of an incompressible fluid [1–9], and work in Fourier wavenumber space, where the degrees of freedom are the Fourier modes $\boldsymbol{u}(\boldsymbol{k}, t)$ of the velocity field as defined in terms of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \boldsymbol{u}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1}$$

In this situation, the main quantity of interest is the energy spectrum, as given by

$$E(k, t) = 2\pi k^2 \langle \boldsymbol{u}(\boldsymbol{k}, t) \cdot \boldsymbol{u}(-\boldsymbol{k}, t) \rangle \tag{2}$$

where $\langle \cdots \rangle$ denotes an ensemble average.

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It is conventional to use an effective viscosity to represent the effect of high-k modes. This has its origins in the work of Heisenberg [10]; but it was Kraichnan [11] who first expressed the eddy viscosity as

$$\delta\nu(k) = -\frac{T_C(k)}{2k^2E(k)} \tag{3}$$

where $T_C(k)$ represents energy transfers due to interactions with unresolved modes. Kraichnan used an analytical turbulence theory—the test-field model—to obtain an eddy viscosity which exhibited the now familiar characteristics of a constant asymptotic value for $k \ll K_C$ and a cusp at $k = K_C$, and this approach inspired much other work of this type. The qualitative form of this eddy viscosity was first established by Domaradzki *et al* [12], who introduced an artificial cutoff into the data obtained from a fully resolved numerical simulation.

We have carried out a numerical simulation with a resolution of $N = 256$ and at a Taylor–Reynolds number of $R_\lambda \approx 190$. For this simulation we chose the fluid kinematic viscosity to be $\nu = 10^{-3}$ with dissipation rate $\varepsilon = 0.149$ (in arbitrary units), giving $L_B/L(t) \approx 5$ where the computational box side is L_B and the integral length scale is L; and $K_{\max}/k_d \approx 1.2$, where k_d is the Kolmogorov dissipation wavenumber. These values are reasonably well in line with current practice [2]. The forcing, which is necessary to maintain a steady state, takes the following form:

$$\boldsymbol{f}(\boldsymbol{k},t) = \begin{cases} \varepsilon \boldsymbol{u}(\boldsymbol{k},t)/(2E_f(t)) & \text{if} \quad 0 < k < k_f \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $E_f(t)$ is the energy contained within the forced modes and $k_f = 1.5$. This forcing scheme is the same as that used by Machiels [13].

The idea underlying the procedure introduced in this paper may be explained by first considering what happens in a truncated simulation where the maximum wavenumber is significantly less than the dissipation wavenumber. As is well known, one of the most obvious effects of such a truncation is an upturn at the high-wavenumber end of the energy spectrum, corresponding to a local build up of energy. This is illustrated in figure 1 where we have plotted an energy spectrum taken from a truncated (i.e. unresolved) simulation after several integration steps. Our aim is to locate this upturn and to correct it in some way. In general terms, the proposed strategy may be described as follows. First, we identify the onset of the upturn with the minimum of the derivative, $\mathrm{d}(\ln E)/\mathrm{d}(\ln k)$, and denote the corresponding wavenumber by $k = k_{\text{upturn}}$. Second, we use the value of this derivative at k_{upturn} to generate a corrected energy spectrum by extrapolating forward from this point. The intended result of this operation is shown in figure 1 as a dashed line. We note that, although our present method is believed to be new, the idea of conducting such direct experiments on a numerical simulation is now of growing interest [14–19]. In particular, there is quite a close similarity of approach between our method and the 'constrained Euler' method of She and Jackson [14], and we shall enlarge on this when we come to the discussion of results.

The following algorithm was carried out after each time-integration step:

(1) A smoothed spectrum E_S is obtained by fitting a polynomial in $\ln k$ to $\ln E$, where $E(k)$ is the usual energy spectrum obtained from the velocity field by shell averaging. A fourth-order polynomial was used for this, as it was found that lower orders do not reproduce the upturn, while significantly higher orders follow the spectrum too closely to give adequate smoothing.

(2) The minimum of the derivative, $\mathrm{d}(\ln E_S)/\mathrm{d}(\ln k)$, is obtained analytically.

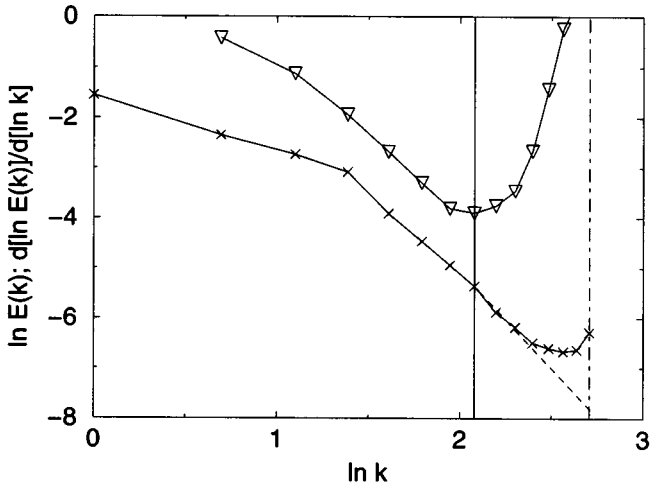

Figure 1. An energy spectrum with an upturn (crosses), its derivative, $\mathrm{d}(\ln E)/\mathrm{d}(\ln k)$ (triangles) and a schematic indication of what the corrected energy spectrum should look like after application of the operational feedback procedure (dashed curve). The vertical solid and dot-dashed lines indicate the positions of k_{upturn} and K_C, respectively.

(3) The gradient, $\mathrm{d}(\ln E_S)/\mathrm{d}(\ln k) = \Gamma$, at $k = k_{\mathrm{upturn}}$ is used to extrapolate the original shell-averaged spectrum, E, forward in wavenumber from k_{upturn} in order to give the corrected spectrum:

$$E_C(k) = \begin{cases} E(k) & \text{if} \quad k \leqslant k_{\mathrm{upturn}} \\ E(k_{\mathrm{upturn}})(k/k_{\mathrm{upturn}})^{\Gamma} & \text{if} \quad k \geqslant k_{\mathrm{upturn}}. \end{cases} \tag{5}$$

(4) The ratio of the corrected spectrum to the shell-averaged spectrum then provides the basis for a correction of the velocity field, thus

$$\boldsymbol{u}_C(\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k})\sqrt{E_C(k)/E(k)}. \tag{6}$$

As a first test for this procedure, we compared three cases: a resolved simulation with $N = 256$, an unresolved simulation with $N = 64$ (i.e. without compensation for the missing modes) and a compensated simulation with $N = 64$ (following the procedure outlined above). All simulations were allowed to run for approximately 24 evolved eddy turnover times.

In figure 2 we have plotted k_{upturn}/K_C against time, showing values corresponding to every 25 integration time steps. We see that it appears to fluctuate around a value of $k_{\mathrm{upturn}}/K_C \approx 0.6$, and we note a period of rapid fluctuations between 12 and 18 eddy turnover times.

Figure 3 shows the evolution with time of the total energy for each of the three simulations. It may be seen that the fluctuations in the case of the unresolved 'control' simulation were larger than in the other two cases, but otherwise there is no significant difference of behaviour. The mean values found by averaging over time, with error estimates given by twice the standard deviation, were: resolved simulation with $N = 256$, $E = 0.90 \pm 0.04$; unresolved simulation with $N = 64$, $E = 0.96 \pm 0.12$; and compensated simulation with $N = 64$, $E = 0.89 \pm 0.03$. Evidently, despite the large number of eddy turnover times, there is no significant difference between the mean levels (the fluctuations are a different matter).

However, when one considers the 'microscopic' aspects, the picture is quite different. Energy spectra, time averaged over the final 15 eddy turnover times of the simulations, are given in figure 4. Here, the problems in the unresolved simulation are clearly seen, with

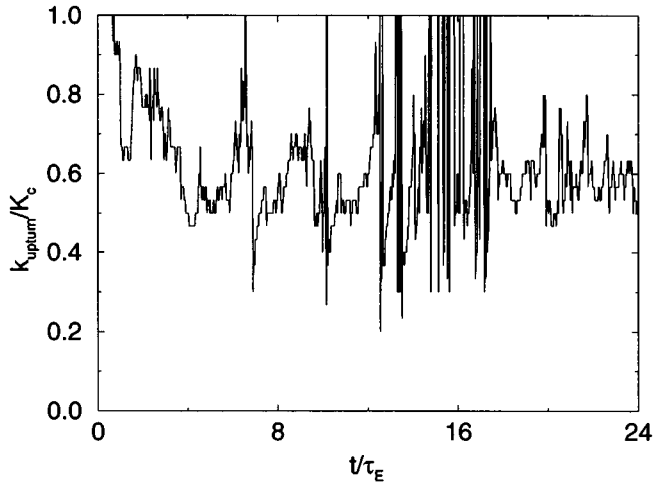

Figure 2. k_{upturn}/K_C plotted as a function of time. A value of $k_{\text{upturn}} = K_C$ indicates that the operational method has not amended the velocity field in any way. The time axis has been scaled on τ_E, the eddy turnover time.

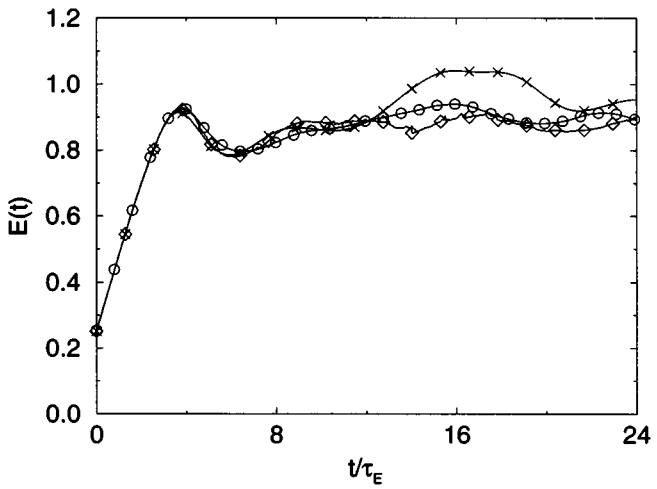

Figure 3. Evolution of total energy showing the results from the resolved 256^3 simulation (circles), the unresolved 64^3 simulation (crosses) and the compensated 64^3 simulation (diamonds).

the upturn dominating the energy spectrum. In contrast, the spectrum obtained from the compensated simulation, shows a good match with that obtained from the resolved simulation.

We have also investigated the effect of this feedback procedure on the velocity derivative skewness $S(t)$. The time-averaged value obtained from the resolved simulation, with an estimate of the error (given by twice the standard deviation) is $S = -0.50 \pm 0.07$ in agreement with experiment [20]. However, as noted by Dubois *et al* [19], the simple act of truncating a velocity field in Fourier space—and hence removing the small scales—will in itself cause a reduction in the skewness and this is inevitable with any variety of LES. Therefore, in order to make a fair comparison with the results of our two 64^3 simulations, we have also computed the skewness based on a number of truncated 256^3 velocity fields. This gave a

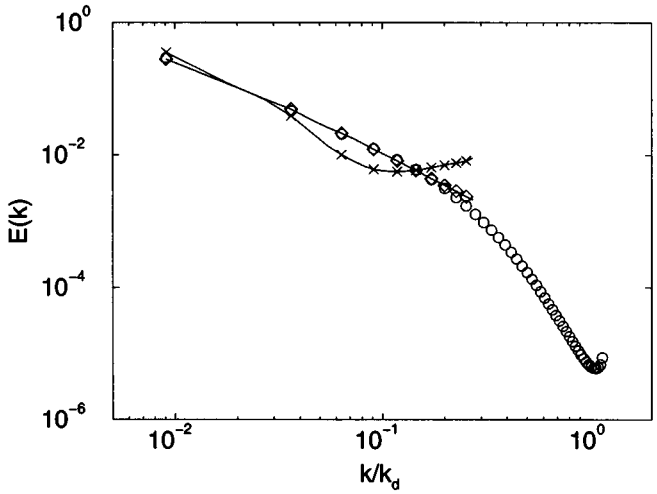

Figure 4. Average evolved energy spectra, showing the results from the resolved 256^3 simulation (circles), the unresolved 64^3 simulation (crosses) and the compensated 64^3 simulation (diamonds).

value of $S = -0.33 \pm 0.04$. The unresolved simulation with $N = 64$ gave a value of $S = -0.12 \pm 0.04$, indicating a distribution closer to the Gaussian case than for the resolved 256^3 result, while the compensated simulation gave a value of $S = -0.30 \pm 0.05$ in agreement with the result obtained from the truncated 256^3 fields. Similar results were found by She and Jackson [14], who obtained $S = 0.31 \pm 0.03$ from a truncation of a fully resolved simulation and $S = 0.28 \pm 0.03$ from their 'constrained Euler' simulation.

It is possible to interpret the compensated simulation presented here in terms of the usual spectral LES by generating an effective eddy viscosity, $\delta\nu(k, t_n)$, at each time step, which would give an enhanced dissipation rate equivalent to the rate of energy removal due to the operational procedure. It is readily shown that in going from t_n to t_{n+1} this takes the form:

$$\delta\nu(k, t_n) \equiv \frac{E(k, t_{n+1}) - E_C(k, t_{n+1})}{2k^2 \delta t\, E(k, t_n)}. \tag{7}$$

The time-averaged result is plotted in figure 5 alongside an empirical eddy viscosity, computed from a resolved velocity field in the same way as by Domaradzki *et al* [12].

The significance of such agreement is that it establishes the localness in wavenumber space of the relevant interactions, in momentum transfer as well as energy transfer. It is, of course, quite usual to *assume* that such interactions are local. For instance, Domaradzki *et al* [12], refer twice to their results as indicating the importance of local interactions. Yet in principle both the momentum (Navier–Stokes) equation and the energy conservation equations are highly nonlocal in wavenumber. In this work, a direct action on the velocity field at a given wavenumber produces an effective viscosity at that wavenumber but affects the spectrum nowhere else. Thus the relevant nonlinear transfers are indeed local.

It is also worth noting that some implementations of the operational procedure resulted in a net energy gain and this would imply occasional negative values for the instantaneous viscosity. However, it may be seen that the time-averaged effective viscosity is generally positive, except at wavenumbers where there was forcing.

We note that the 'constrained Euler' simulation [14] could also be interpreted in terms of such a viscosity (although as no results for this were given, we are unable to make a detailed comparison) and that these authors concluded that their simulation was equivalent to a LES

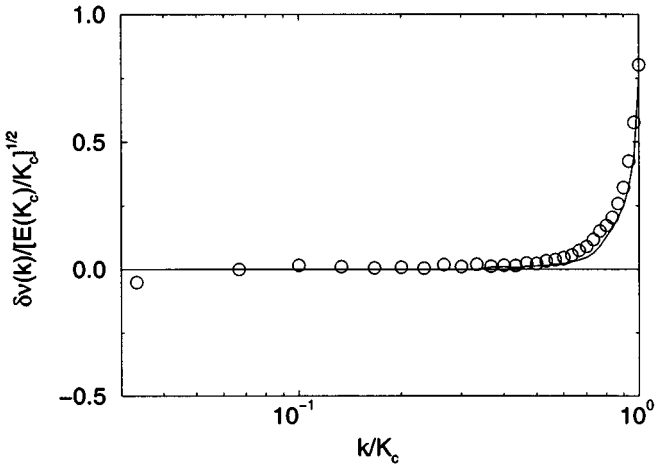

Figure 5. Average equivalent eddy viscosity computed from the compensated 64^3 simulation (curve) compared with the 'empirical' eddy viscosity computed from the resolved 256^3 simulation (circles).

of the Navier–Stokes equations. There are various detailed differences between our method and that of She and Jackson [14]. For instance, we constrain modes only where deviation is noted, which in practice is always near the cutoff wavenumber, whereas they constrain all resolved modes. However, the similarity of approach and (so far as we can tell) results is very interesting and we hope to elucidate these matters in further work.

The operational procedure outlined here gives interesting results at the relatively low Reynolds number we have explored. However, it must be borne in mind that it depends on an assumption about the form taken by the spectrum, if we truncate the high-wavenumber modes. Under the present restricted circumstances, there is only one possible outcome. In the absence of nonlinear transfer to higher-k modes, the energy must increase at the cutoff wavenumber. This guarantees the stability of the feedback process. However, we should remind ourselves that although this is true for the simple spectral method used here, and for isotropic turbulence, it is not necessarily true for more realistic flows. This is a matter which would require further investigation.

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